Writing Position Vectors in 3-d Space: A Student Difficulty With Spherical Unit Vectors in Intermediate E&M

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Abstract. An intermediate E&M course (i.e. based on Griffiths [1]) involves the extensive integration of vector calculus concepts and notation with abstract physics concepts like field and potential. We hope that students take what they have learned in their math courses and apply it to help represent and make sense of the physics. To assess how well students are able to do this integration and application I have developed several simple concept tests on position and unit vectors in non-Cartesian coordinate systems as they are used in intermediate E&M. In this paper I describe one of these concept tests and present results that show both undergraduate physics majors and physics graduate students have difficulty using spherical unit vectors to write position vectors in 3-d space.

Keywords: spherical unit vectors, position vector, intermediate electrodynamics, Maxwell's integral equations
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INTRODUCTION

This paper presents results from a study of student understanding of spherical unit vectors in an intermediate E&M course (i.e. based on Griffiths [1]). Students were asked to write position vectors in terms of spherical unit vectors for several carefully chosen points in 3-d space. They displayed many different difficulties that will be described in this paper. The ability to write position vectors in this fashion is an important skill in an intermediate E&M course since it is used extensively when calculating electric and magnetic fields and potentials for spherically symmetric continuous charge and current distributions.

This study was motivated by the observation that although students in my intermediate E&M courses were very competent using spherical *coordinates* in many different contexts, I had an intuitive feeling that they still didn't quite seem to "get" spherical *unit vectors*. I wanted to figure out what in particular was confusing them.

THE UBIQUITOUS POSITION VECTOR

An important assumption of this study was that after taking a course in intermediate E&M, students should have a functional understanding of the mathematical meaning of Maxwell's integral equations. That is, they should be able to correctly set up the appropriate integral equation given an arbitrary continuous charge or current distribution.

There are many different Maxwell's integral equations: for 1-d, 2-d, and 3-d continuous charge and current distributions, for electric field and potential, for magnetic field and potential, for vacuum, for dielectric, etc. For illustrative purposes, I show just two here. Following Griffith's notation, Maxwell's integral equations in vacuum for electric potential in 1-d and electric field in 2-d are the following:

$$V(\bar{r}) = \tfrac{1}{4\pi\varepsilon_o} \int_C \frac{\lambda(\bar{r}')\,dl'}{|\bar{r}-\bar{r}'|^2} \tag{1}$$

$$\bar{E}(\bar{r}) = \tfrac{1}{4\pi\varepsilon_o} \iint_S \frac{\bar{r}-\bar{r}'}{|\bar{r}-\bar{r}'|} \cdot \frac{\sigma(\bar{r}')\,da'}{|\bar{r}-\bar{r}'|^2} \tag{2}$$

where:

$\bar{r}$ is a position vector that indicates the point in 3-d space where the potential or field is to be determined, and

$\bar{r}'$ is a position vector that gives the points in 3-d space where the *source* charges are located.

The integration is over a continuous *source* charge distribution (given by position vector $\bar{r}'$) modeled as:

C, a 1-d line of point particles, with linear charge density λ, or

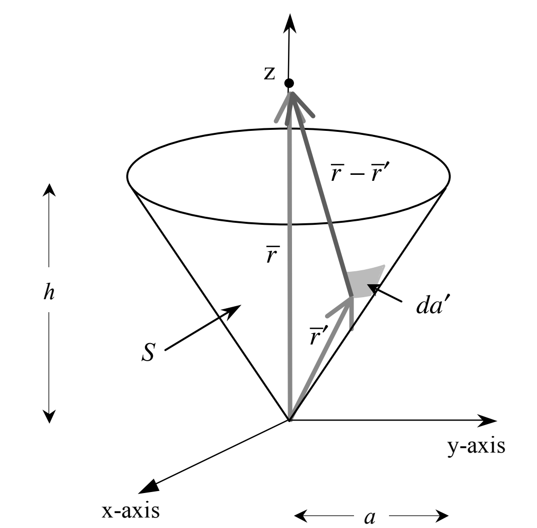
S, a 2-d surface of point patches, with surface charge density σ.

Notice how ubiquitous position vectors $\bar{r}$ and $\bar{r}'$ are in equations (1) and (2). They appear in multiple places and are central to the calculation. The equations for magnetic field and potential and for 3-d also depend extensively on the two position vectors.

Hence a functional understanding of Maxwell's integral equations necessarily entails a functional understanding of $\bar{r}$ and $\bar{r}'$ (i.e. the ability to write down $\bar{r}$ and $\bar{r}'$ for any given charge or current distribution). So, given the text shown in Fig. 1a, and the diagram shown in Fig. 1b, students with a functional understanding of $\bar{r}$ and $\bar{r}'$ should be able to produce the appropriate definitions for all variables as shown in Fig. 1c.

(a) A hollow cone of radius a, and height h, is centered on the z-axis with its tip at the origin and it's base in the + z direction. It has uniform surface charge density, σ, on its curved sides, but no charge on its base. Find the electric potential at a point on the *z*-axis above the cone.

(b)



(c) da' = generic patch of source charge

$\bar{r}$ = field point $\quad\quad \bar{r}'$ = points to patch

$= z\hat{z} \quad\quad = r'\hat{r}'$

$$\hat{r}' = \sin\theta' \cos\phi' \,\hat{x} + \sin\theta' \sin\phi' \,\hat{y} + \cos\theta' \,\hat{z}$$

$$\left\{\begin{array}{l} 0 < r' < \sqrt{a^2+h^2} \\ 0 < \phi' < 2\pi \\ \quad \theta' = \tan^{-1}\left(a/h\right) \end{array}\right\}$$

FIGURE 1. (a) Text of problem statement. (b) Well-labeled diagram of problem statement. (c) Variable definitions a student should produce to accompany (b).

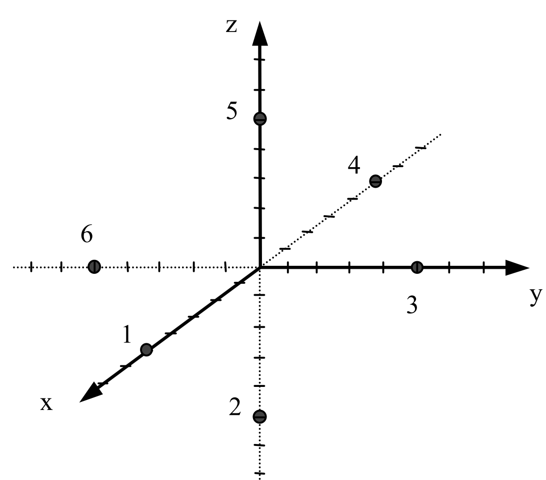
SPHERICAL UNIT VECTOR CONCEPT TEST

The concept test shown in Fig. 2 was developed at Drury University over a period of four years. Each year, it was given to the students in our intermediate E&M course. Based on the results for a given year, it was modified to increase its diagnostic capability and given again the next year until it reached the final form shown. The paper-and-pencil version of the test typically takes 10-15 minutes. In videotaped interviews students have taken anywhere from 10-45 minutes depending on how much they try to derive from scratch or how confused they are.

DATA COLLECTION

I report here the results from giving this concept test to a total of 46 physics majors at three different colleges and universities. 42 students took the paper-and-pencil test. In addition, 4 students (from LP-ug) were videotaped working through this problem. When they had finished they were asked some follow-up and clarifying questions about their reasoning in certain places. Videotaping was done to get some insight into the student reasoning process because on the paper-and-pencil test students don't write explanations for why they wrote what they did.

Please write $\bar{r}$ in terms of $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ for the following six different points. Show all work.



(a)

$\bar{r}_1 = 5\hat{r} \quad \bar{r}_2 = 5\hat{r} \quad \bar{r}_3 = 5\hat{r}$

$\bar{r}_4 = 5\hat{r} \quad \bar{r}_5 = 5\hat{r} \quad \bar{r}_6 = 5\hat{r}$

(b)

FIGURE 2. (a) Concept test used to probe student understanding of spherical unit vectors. (b) Expected answer.

TABLE 1. Schools from which data was collected and details of how the concept test was given at each school.

Institution	Textbook	N[a]	How Given	When Given
Small private liberal arts college in the upper Midwest, PLA	Pollack & Stump [2]	12 of 12	As homework for credit	After completing both Chp2 (Vector Calculus) in class and relevant homework from Chp2
Small public university in the upper Midwest, SP	Griffiths	6 of 6	Quiz	After completing both lecture on Section 1.4 (curvilinear coordinates) and relevant homework
Large public university in the Southwest, undergraduates LP-ug	Griffiths	14 of 26	Volunteers who stayed after class	During the last week of a full year of intermediate E&M
Large public university in the Southwest, graduate students LP-g	n/a	14 of 21	Volunteers who stayed after class	During the last week of the first year of graduate school, in their quantum course

[a] Indicates how many of the students who were officially registered for the course actually took the concept test.

A list of the schools that participated is shown in Table 1, along with details of how the concept test was given in each case. For PLA and SP, the instructor gave the concept test to their students. For LP-ug and LP-g, I gave the concept test to volunteers who agreed to take it. About three quarters of the undergraduates (PLA, SP, LP-ug) were juniors and a quarter were seniors. There were eleven females and thirty-five males.

The design of this study was challenging because, as Table 1 shows, the number of students enrolled in a course in intermediate E&M is rather low. So it was difficult to get a sufficient sample-size for the population in question. That is why several different schools and levels of student were included.

A further difficulty with the study was that for LP-ug and LP-g, only about half and two-thirds, respectfully, of those registered for the course actually took the concept test. We would typically expect that students who chose to take such an optional test are of higher caliber than those who do not, and thus they should perform better than those who did not take the concept test. So the low performance of the students in LP-g and LP-ug provides an upper ceiling for each class's response as a whole.

RESULTS AND DISCUSSION

Table 2 shows the six most common student answers to the concept test shown in Fig. 2. Only answers for the position vector for point 1 are shown since all students were consistent in the form they used for the six different position vectors.

Note that none of the six most common student answers match the expected answer (Fig. 2b). I have arbitrarily labeled the first three errors A1, A2, A3. I grouped them together as "A" type errors because in all three cases students explicitly wrote the spherical unit vectors $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ as part of their answer. "B" type errors were grouped together because these answers explicitly referenced just the spherical *coordinates* themselves for the point of interest. They had no *unit vectors* explicitly written. "C" type errors were of a different kind than "A" or "B" type.

It is possible that in students' minds "A" and "B" type errors actually have the same meaning but just use different notation. But, we stress, for computational reasons that we don't have space to elaborate upon here, correct notation is crucial to calculating correctly with equations like (1) and (2). Hence, the emphasis in our classification upon whether students did or did not explicitly include unit vectors in their answers.

It's possible that error A1 comes from pattern matching to the Cartesian case. That is, an ordered triple in Cartesian coordinates is equivalent to a linear combination of its coordinate values multiplied by its unit vectors:

$$(x, y, z) = x\,\hat{x} + y\,\hat{y} + z\,\hat{z} \tag{3}$$

Unfortunately, it is completely false for spherical

TABLE 2. Six most common student answers for $\bar{r}_1$

Student Answer	Error Label
$5\hat{r} + \frac{\pi}{2}\hat{\theta} + 0\hat{\phi}$	A1
$5\hat{r},\ \frac{\pi}{2}\hat{\theta},\ 0\hat{\phi}$	A2
$\left(5\hat{r},\ \frac{\pi}{2}\hat{\theta},\ 0\hat{\phi}\right)$	A3
$\left(5,\ \frac{\pi}{2},\ 0\right)$	B1
$r = 5, \theta = \frac{\pi}{2}, \phi = 0$	B2
$(5\sin\theta\cos\phi)$	C

TABLE 3. Results for each school and composite totals for the concept test shown in Fig 2

Institution	N	Answer[b] Correct	A1	A2	A3	B1	B2	C	Other	r, θ, ϕ all correct[c]
PLA	12	0	6	2	1	2	1	0	0	6
SP	6	0	3	0	1	1	0	1	0	3
LP-ug	14	1	6	3	0	1	2	0	1	6
LP-g	14	0	6	1	1	4	0	2	0	7
	46 (100%)	1 (2%)	21 (46%)	6 (13%)	3 (7%)	8 (17%)	3 (7%)	3 (7%)	1 (2%)	22 (48%)

[b] Number of students at each school who made this kind of error. See Table 2 for defintions of symbols.
[c] Number of students at each school who got r, θ, and ϕ values correct for all six points.

coordinates:

$$(r,\theta,\phi) \neq r\,\hat{r} + \theta\hat{\theta} + \phi\hat{\phi} \tag{4}$$

No one who put answer A1 on the paper-and-pencil test explained why they wrote what they did. But in video interviews at LP-ug, two of the four students agreed they were pattern matching to the Cartesian case. At least one explicitly said this was because he understood Cartesian coordinates best.

It's possible that students who answered A2 and A3 were also doing pattern matching, along the lines of equations (3) and (4). Although unlike those equations, A2 and A3 explicitly have unit vectors written as part of the 3-tuple. Perhaps those students were doing some kind of blending between the left and right hand sides of equation (3). That is, perhaps they were thinking:

$$(x,y,z) \;=\; x\hat{x} + y\hat{y} + z\hat{z} \;=\; (x\hat{x}, y\hat{y}, z\hat{z}) \tag{5}$$

Although the evidence and context is suggestive, it is hard to say without more explanation from the students about why they answered the way they did.

Table 3 shows the results for each school as well as a composite total for all schools. Some results are quite striking. For example, only one person out of all forty-six got the correct answer. No graduate students got the right answer. Nearly half of all students put answer A1. Almost a fifth listed the spherical coordinates. Seventy percent (correct+A1+A2+A3 +other) explicitly used $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ in their answer, while thirty percent completely left off any mention of unit vectors at all, even though the problem statement explicitly asked them to do so (Fig. 2a). Lastly, slightly less than half of all students were able to correctly write down all eighteen values of r, θ, and ϕ for the six points. The most common mistakes were for the values of ϕ (67% correct) and θ (63% correct). Also, 20% of students made reversal mistakes, where they gave the ϕ value for θ and vice-versa.

Despite the small sample size for each school, we claim the composite results indicate that three measures can be considered to be reliable when just looking at the results for an individual school. First, essentially no one got the right answer. As discussed earlier, those answering with A-type answers were most likely pattern matching and unsure about the connection between 3-tuples and unit vectors. It's harder to make conclusions about students who gave B-type answers, since they left off any mention of unit vectors entirely. Did they misread the question? Or did they not understand what the question was asking for? Or did they perhaps think that unit vectors are implicit in the notation they used?

Second, all schools individually had about 50% of their students answering A1, which matches the composite of 46%. Clearly answer A1 was consistently the most common answer at each school. And third, all individual schools had about 50% of students able to get all eighteen values of r, θ, and ϕ correct.

Lastly, the take home message is four-fold. First, virtually no student was able to get the correct response. This means we need to seriously rethink our instruction on this topic. Second, there seem to be two major classes of notational error (the A-type and B-type) and a much more rare third class of error (the C-type). Third, there don't seem to be really big differences between undergraduates and graduate students. And fourth, the results don't seem to depend on different texts, different instructors, different schools, different class sizes, and different years in school.

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